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THE AXISYMMETRIC DOUBLE CONTACT PROBLEM FOR A FRICTIONLESS ELASTIC LAYER†

M. B. CIVELEK and F. ERDOGAN
Lehigh University, Bethlehem, Pennsylvania, 18105, U.S.A.

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Abstract—The general axisymmetric double contact problem for an elastic layer pressed against a half space by an elastic stamp is considered. The problem is solved under the assumptions that the three materials have different elastic properties, the contact along the interfaces is frictionless and only compressive normal tractions can be transmitted across the interfaces, and, in the case of the elastic stamp, the local radius of curvature of the stamp is large compared to the stamp-layer contact radius. The problem is reduced to a system of singular integral equations in which the contact pressures are the unknown functions. The solution is obtained and extensive numerical results are given for three stamp geometries, namely, rigid and elastic spherical stamps, and a flat-ended rigid cylindrical stamp. The results show that in the case of a flat-ended rigid cylindrical stamp the radius b of the contact area between the layer and the subspace is independent of the magnitude P of the total transmitted load and in all other cases b will depend on P.

1. INTRODUCTION

Because of its application to a great variety of important structures of practical interest (such as foundations, pavements in roads and runways, rolling mills, ball and roller bearings, and other structures consisting of layered media), in the past the contact problem in solid mechanics involving an elastic layer and/or an elastic half space has been very widely studied. The general description of the problem may be found, for example, in[1–4]. Some of the typical solutions given in recent years may be found in[4–9] where it is assumed that the contact between the elastic layer and the subspace is either one of perfect adhesion or frictionless with the additional (rather unrealistic) condition that across the interface the normal component of the displacement vectors is continuous. The problem of a frictionless elastic layer on an elastic foundation where it was assumed that the contact stresses can only be compressive was discussed in[10–13]. In[11] the plane and the axisymmetric problems are solved with a concentrated load or a uniform pressure applied to a certain portion of the layer surface. The solution of the plane contact problem in which the load is applied to the layer through a rigid stamp with a cylindrical or rectangular profile is given in[12]. The similar problem in which the elastic layer is approximated by a thin plate with a certain bending stiffness and is subjected to normal loads on its free surface was discussed in[14–17].

The results of the studies on this so-called "receding contact problem" which has been considered in[10–17] indicate that the contact area between the layer and the subspace depends only on the relative distribution of the applied load and is independent of the load amplitude. However, in[12] it was shown that this may not be the case if the load is applied to the layer through a stamp. In this double contact problem there are two unknown

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functions, namely, the pressures between the stamp and the layer p_1 and the layer and the subspace p_2. If the load is applied to the layer directly or, more generally, through a rectangular stamp (the flat end of which remains parallel to the layer surface while the load is applied), actually the unknown functions are p_i/P, $(i = 1, 2)$ where the constant P is a measure of the load amplitude (say, the resultant force). Hence, P appears in the results as a multiplicative constant in the contact pressures only. On the other hand, if the rectangular stamp has any other profile (with a nonvanishing first derivative), as shown in[12] the layer subspace contact area will depend on the load amplitude P.

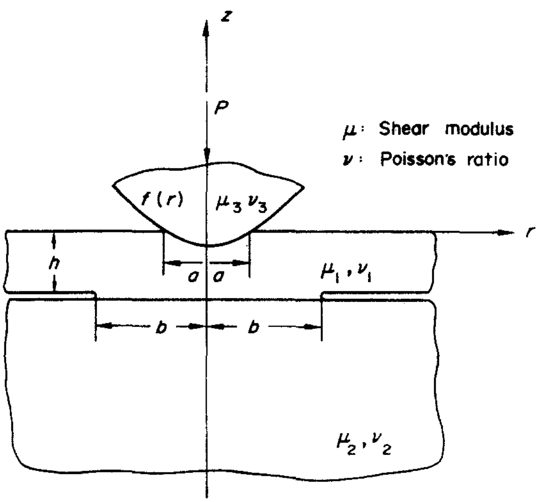

Fig. 1. The Stamp-layer-half space geometry.

In this paper we will consider the general axisymmetric double contact problem for three different elastic materials, namely, a semi-infinite subspace, a layer of finite thickness, and a stamp (Fig. 1). Detailed results will also be given for a rigid stamp with a spherical or a flat-ended cylindrical surface. The elastic (soft) stamp solution may find its application, for example, in load transmission problems in pavements. The problem of a spherical stamp on a frictionless elastic layer has also been studied experimentally by using the photoelastic stress freezing technique, and the results were given in a recent paper[18].

2. DERIVATION OF THE INTEGRAL EQUATIONS

Consider the axisymmetric double contact problem for three different elastic materials shown in Fig. 1. Let μ_i, ν_i, $(i = 1, 2, 3)$ be the elastic constants of the layer l, the half space 2, and the stamp 3. The problem will be solved under the assumptions that, (a) the contact along the interfaces is frictionless and only compressive normal tractions can be transmitted across the contact surfaces, and (b) the local radius of curvature R of the elastic stamp is sufficiently large compared to the radius a of the stamp-layer contact area so that in expressing the surface displacement of the stamp in the contact area in terms of the contact pressure p_1, the medium 3 may be approximated by an elastic half space. The assumption (a) appears to be realistic. If we are not interested in the global stress distribution in the stamp and if the external loads are applied to the stamp at a location sufficiently far from the contact region, the error introduced through the assumption (b) would not be expected to be very significant[1, 3]. The governing equations of the problem then are (see Fig. 1)

$$(\lambda_i + 2\mu_i)\left(\frac{\partial^2 u_i}{\partial r^2} + \frac{1}{r}\frac{\partial u_i}{\partial r} - \frac{u_i}{r^2} + \frac{\partial^2 w_i}{\partial r\,\partial z}\right) + \mu_i\left(\frac{\partial^2 u_i}{\partial z^2} - \frac{\partial^2 w_i}{\partial r\,\partial z}\right) = 0,$$

$$(\lambda_i + 2\mu_i)\left(\frac{\partial^2 u_i}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial u_i}{\partial z} + \frac{\partial^2 w_i}{\partial z^2}\right) - \frac{\mu_i}{r}\left(\frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial r}\right) - \mu_i\left(\frac{\partial^2 u_i}{\partial r\,\partial z} - \frac{\partial^2 w_i}{\partial r^2}\right) = 0, \qquad (i = 1, 2, 3); \tag{1a,b}$$

$$\sigma_{ir} = (\lambda_i + 2\mu_i)\frac{\partial u_i}{\partial r} + \lambda_i\left(\frac{u_i}{r} + \frac{\partial w_i}{\partial z}\right),$$

$$\sigma_{i\theta} = (\lambda_i + 2\mu_i)\frac{u_i}{r} + \lambda_i\left(\frac{\partial u_i}{\partial r} + \frac{\partial w_i}{\partial z}\right),$$

$$\sigma_{iz} = (\lambda_i + 2\mu_i)\frac{\partial w_i}{\partial z} + \lambda_i\left(\frac{\partial u_i}{\partial r} + \frac{u_i}{r}\right),$$

$$\tau_{irz} = \mu_i\left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial r}\right), \qquad (i = 1, 2, 3); \tag{2a–d}$$

subject to the following boundary conditions:

$$\tau_{1rz} = 0, \qquad \tau_{3rz} = 0, \qquad \sigma_{1z} = \sigma_{3z}, \qquad (z = 0,\ r \geq 0) \tag{3a–c}$$

$$\left.\begin{aligned} \sigma_{1z} &= 0, && (z = 0,\ r > a), \\ w_3 - w_1 &= f(r), && (z = 0,\ 0 \leq r < a), \end{aligned}\right\} \tag{4a,b}$$

$$\tau_{1rz} = 0, \qquad \tau_{2rz} = 0, \qquad \sigma_{1z} = \sigma_{2z}, \qquad (z = -h,\ r \geq 0) \tag{5a–c}$$

$$\left.\begin{aligned} \sigma_{1z} &= 0, && (z = -h,\ r > b), \\ w_1 - w_2 &= 0, && (z = -h,\ 0 \leq r < b). \end{aligned}\right\} \tag{6a,b}$$

where $f(r)$ is a known function obtained from the equation giving the profile of the stamp, h is the thickness of the layer, a is the radius of the contact area between the stamp and the layer, and b is the radius of that between the layer and the subspace. b and, in the case of a stamp with rounded corners, a are unknown constants.

Using, for example, Hankel transforms the solution of (1) and the relevant stress components may be expressed as

$$u_1(r, z) = \int_0^\infty [(A_1 + zA_2)e^{-\alpha z} + (A_3 + zA_4)e^{\alpha z}]J_1(\alpha r)\alpha\, d\alpha,$$

$$w_1(r, z) = \int_0^\infty \left\{\left[A_1 + \left(\frac{\kappa_1}{\alpha} + z\right)A_2\right]e^{-\alpha z} + \left[-A_3 + \left(\frac{\kappa_1}{\alpha} - z\right)A_4\right]e^{\alpha z}\right\}J_0(\alpha r)\alpha\, d\alpha; \tag{7a,b}$$

$$u_2(r, z) = \int_0^\infty (A_5 + zA_6)e^{\alpha z}J_1(\alpha r)\alpha\, d\alpha,$$

$$w_2(r, z) = \int_0^\infty \left[-A_5 + \left(\frac{\kappa_2}{\alpha} - z\right)A_6\right]e^{\alpha z}J_0(\alpha r)\alpha\, d\alpha; \tag{8a,b}$$

$$u_3(r, z) = \int_0^\infty (A_7 + zA_8)e^{-\alpha z}J_1(\alpha r)\alpha \, d\alpha,$$

$$w_3(r, z) = \int_0^\infty \left[A_7 + \left(\frac{\kappa_3}{\alpha} + z\right)A_8\right]e^{-\alpha z}J_0(\alpha r)\alpha \, d\alpha; \qquad (9a, b)$$

$$\frac{1}{2\mu_1}\sigma_{z1} = \int_0^\infty \{-[\alpha(A_1 + zA_2) + 2(1-\nu_1)A_2]e^{-\alpha z} + [-\alpha(A_3 + zA_4) + 2(1-\nu_1)A_4]e^{\alpha z}\}J_0(\alpha r)\alpha \, d\alpha,$$

$$\frac{1}{2\mu_1}\tau_{1rz} = \int_0^\infty \{-[\alpha(A_1 + A_2 z) + (1-2\nu_1)A_2]e^{-\alpha z} + [\alpha(A_3 + zA_4) - (1-2\nu_1)A_4]e^{\alpha z}\}J_1(\alpha r)\alpha \, d\alpha; \qquad (10a,b)$$

$$\frac{1}{2\mu_2}\sigma_{2z} = \int_0^\infty [-\alpha(A_5 + zA_6) + 2(1-\nu_2)A_6]e^{\alpha z}J_0(\alpha r)\alpha \, d\alpha,$$

$$\frac{1}{2\mu_2}\tau_{2rz} = \int_0^\infty [\alpha(A_5 + zA_6) - (1-2\nu_2)A_6]e^{\alpha z}J_1(\alpha r)\alpha \, d\alpha; \qquad (11a,b)$$

$$\frac{1}{2\mu_3}\sigma_{3z} = \int_0^\infty [-\alpha(A_7 + zA_8) - 2(1-\nu_3)A_8]e^{-\alpha z}J_0(\alpha r)\alpha \, d\alpha,$$

$$\frac{1}{2\mu_3}\tau_{3rz} = \int_0^\infty [-\alpha(A_7 + zA_8) - (1-2\nu_3)A_8]e^{-\alpha z}J_1(\alpha r)\alpha \, d\alpha; \qquad (12a,b)$$

where $\kappa_i = 3 - 4\nu_i$, $(i = 1, 2, 3)$. $A_1, \ldots, A_8$ are (unknown) functions of the integration variable α, and are obtained from the boundary conditions (3–6). Equations (3) and (5) give six homogeneous algebraic equations in A_k and may be used to eliminate six of the unknowns. The remaining two functions may be obtained from the system of dual integral equations resulting from the mixed conditions (4) and (6). A more direct approach to obtain the integral equations of the problem would be the following: First define

$$p_1(r) = -\sigma_{1z}(r, 0), \qquad p_2(r) = -\sigma_{1z}(r, -h). \qquad (13a,b)$$

From (4a) and (6a) it follows that

$$p_1(r) = 0, \quad (r > a); \qquad p_2(r) = 0, \quad (r > b), \qquad (14a,b)$$

and on the contact areas the pressures p_1 and p_2 are unknown. By replacing (4) and (6) by (13) and (14), the unknown functions $A_1, \ldots, A_8$ may be expressed in terms of p_1 and p_2 as

$$A_i(\alpha) = m_{i1}(\alpha)\int_0^a p_1(\rho)J_0(\alpha\rho)\rho \, d\rho + m_{i2}(\alpha)\int_0^b p_2(\rho)J_0(\alpha\rho)\rho \, d\rho, \qquad (i = 1, \ldots, 8), \quad (15)$$

where the known functions $m_{ij}(\alpha)$, $(i = 1, \ldots, 8, j = 1, 2)$ are given in the Appendix. The solution (7–12) with A_i as given by (15) satisfies all of the boundary conditions stated by (3–6) except the mixed conditions (4b) and (6b), which, by using (7b), (8b), (9b) and (15), give the following pair of integral equations in the unknown functions p_1 and p_2:

$$
\begin{aligned}
&(\gamma_1+\gamma_3)\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\quad-2\gamma_1\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty \frac{2\alpha^2h^2+2\alpha h+1-\mathrm{e}^{-2\alpha h}}{4\alpha^2h^2+2-\mathrm{e}^{-2\alpha h}-\mathrm{e}^{2\alpha h}}J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\quad-2\gamma_1\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty \frac{(1-\alpha h)\mathrm{e}^{-\alpha h}-(1+\alpha h)\mathrm{e}^{\alpha h}}{4\alpha^2h^2+2-\mathrm{e}^{-2\alpha h}-\mathrm{e}^{2\alpha h}}J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\qquad=f(r),\qquad(0\le r<a),\\
&(\gamma_1+\gamma_2)\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho\int_0^\infty J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\quad-2\gamma_1\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho\int_0^\infty \frac{2\alpha^2h^2+2\alpha h+1-\mathrm{e}^{-2\alpha h}}{4\alpha^2h^2+2-\mathrm{e}^{-2\alpha h}-\mathrm{e}^{2\alpha h}}J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\quad-2\gamma_1\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty \frac{(1-\alpha h)\mathrm{e}^{-\alpha h}-(1+\alpha h)\mathrm{e}^{\alpha h}}{4\alpha^2h^2+2-\mathrm{e}^{-2\alpha h}-\mathrm{e}^{2\alpha h}}J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha\\
&\qquad=0,\ (0\le r<b),\qquad\text{(16a,b)}
\end{aligned}
$$

where

$$\gamma_i=(1-\nu_i)/\mu_i,\qquad(i=1,2,3).\tag{17}$$

Noting that

$$\int_0^\infty J_0(\alpha r)J_0(\alpha\rho)\,\mathrm{d}\alpha=\begin{cases}\dfrac{2}{\pi r}K\left(\dfrac{\rho}{r}\right), & (0\le\rho<r),\\[2ex] \dfrac{2}{\pi\rho}K\left(\dfrac{r}{\rho}\right), & (r<\rho),\end{cases}\tag{18}$$

(where K is the complete elliptic integral of the first kind) and differentiating (16) with respect to r we obtain

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^a h(r,\rho)\left(\frac{1}{\rho-r}-\frac{1}{\rho+r}\right)p_1(\rho)\,\mathrm{d}\rho+2\gamma\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty F_1(r,\rho,\alpha)\,\mathrm{d}\alpha\\
&\qquad+2\gamma\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho\int_0^\infty F_2(r,\rho,\alpha)\,\mathrm{d}\alpha=\frac{1}{\gamma_1+\gamma_3}\frac{\mathrm{d}f(r)}{\mathrm{d}r},\qquad(0\le r<a),\\
&\frac{1}{\pi}\int_0^b h(r,\rho)\left(\frac{1}{\rho-r}-\frac{1}{\rho+r}\right)p_2(\rho)\,\mathrm{d}\rho+2\beta\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho\int_0^\infty F_1(r,\rho,\alpha)\,\mathrm{d}\alpha\\
&\qquad+2\beta\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\int_0^\infty F_2(r,\rho,\alpha)\,\mathrm{d}\alpha=0,\qquad(0\le r<b),\qquad\text{(19a,b)}
\end{aligned}
$$

where

$$h(r,\rho)=\begin{cases}\dfrac{\rho}{r}E\left(\dfrac{\rho}{r}\right), & (\rho<r),\\[2ex] \dfrac{\rho^2}{r^2}E\left(\dfrac{r}{\rho}\right)-\dfrac{\rho^2-r^2}{r^2}K\left(\dfrac{r}{\rho}\right), & (\rho>r),\end{cases}\tag{20}$$

$$\gamma = \frac{\gamma_1}{\gamma_1 + \gamma_3} = \frac{\dfrac{1-\nu_1}{\mu_1}}{\dfrac{1-\nu_1}{\mu_1} + \dfrac{1-\nu_3}{\mu_3}}, \qquad \beta = \frac{\gamma_1}{\gamma_1 + \gamma_2} = \frac{\dfrac{1-\nu_1}{\mu_1}}{\dfrac{1-\nu_1}{\mu_1} + \dfrac{1-\nu_2}{\mu_2}}, \tag{21}$$

$$F_1(r, \rho, \alpha) = \alpha\left(\frac{2\alpha^2h^2 + 2\alpha h + 1 - e^{-2\alpha h}}{4\alpha^2h^2 + 2 - e^{-2\alpha h} - e^{2\alpha h}}\right)J_1(\alpha r)J_0(\alpha\rho),$$

$$F_2(r, \rho, \alpha) = \alpha\left(\frac{(1-\alpha h)e^{-\alpha h} - (1+\alpha h)e^{\alpha h}}{4\alpha^2h^2 + 2 - e^{-2\alpha h} - e^{2\alpha h}}\right)J_1(\alpha r)J_0(\alpha\rho). \tag{22a,b}$$

Using the symmetry considerations, if we extend the definition of p_1 and p_2 into $(-a, 0)$ and $(-b, 0)$, respectively, in such a way that

$$p_1(r) = p_1(-r), \qquad p_2(r) = p_2(-r), \tag{23}$$

(19) may also be expressed as

$$\frac{1}{\pi}\int_{-a}^{a} \frac{p_1(\rho)}{\rho - r}\,d\rho + \frac{1}{\pi}\int_{-a}^{a} k(r, \rho)p_1(\rho)\,d\rho + \gamma\int_{-a}^{a} p_1(\rho)|\rho|\,d\rho\int_0^\infty F_1(r, \rho, \alpha)\,d\alpha$$

$$+ \gamma\int_{-b}^{b} p_2(\rho)|\rho|\,d\rho\int_0^\infty F_2(r, \rho, \alpha)\,d\alpha = \frac{1}{\gamma_1 + \gamma_3}\frac{df(r)}{dr}, \qquad (-a < r < a),$$

$$\frac{1}{\pi}\int_{-b}^{b} \frac{p_2(\rho)}{\rho - r}\,d\rho + \frac{1}{\pi}\int_{-b}^{b} k(r, \rho)p_2(\rho)\,d\rho + \beta\int_{-b}^{b} p_2(\rho)|\rho|\,d\rho\int_0^\infty F_1(r, \rho, \alpha)\,d\alpha$$

$$+ \beta\int_{-a}^{a} p_1(\rho)|\rho|\,d\rho\int_0^\infty F_2(r, \rho, \alpha)\,d\alpha = 0, \qquad (-b < r < b), \tag{24a,b}$$

where (noting that $h(r, r) = 1$)

$$k(r, \rho) = \frac{h_1(r, \rho) - 1}{\rho - r},$$

$$h_1(r, \rho) = \begin{cases} \left|\dfrac{\rho}{r}\right| E\left(\left|\dfrac{\rho}{r}\right|\right), & (|\rho| < |r|), \\[2ex] \dfrac{\rho^2}{r^2} E\left(\left|\dfrac{r}{\rho}\right|\right) - \dfrac{\rho^2 - r^2}{r^2} K\left(\left|\dfrac{r}{\rho}\right|\right), & (|\rho| > |r|). \end{cases} \tag{25a,b}$$

The dominant part of the integral equations (24) has a simple Cauchy type singularity. However, in order to avoid some convergence difficulties encountered in the numerical analysis, it is worthwhile to examine the Fredholm type kernels in the integral equations somewhat more closely. In (25a), observing that

$$\lim_{\rho \to r} k(r, \rho) = \left[\frac{d}{d\rho} h_1(r, \rho)\right]_{\rho \to r},$$

using (25b), it can be shown that for small values of $|\rho - r|$ we have

$$k(r, \rho) = \frac{1}{2r}\log|\rho - r| + 0(1). \tag{26}$$

Also, from (22) it is seen that, as $\alpha \to 0$, F_1 and F_2 behave as α^{-1}. Hence, considered separately, the Fredholm kernels in (24) involving the integrals of F_1 and F_2 are not bounded. On the other hand, writing

$$\int_0^\infty F_j(r, \rho, \alpha)\,\mathrm{d}\alpha = \int_0^\varepsilon F_j(r, \rho, \alpha)\,\mathrm{d}\alpha + \int_\varepsilon^\infty F_j(r, \rho, \alpha)\,\mathrm{d}\alpha, \tag{27}$$

assuming ε to be very small, and hence replacing F_j by its asymptotic expansion around $\alpha = 0$, for example (19b) may be expressed as

$$\begin{aligned}
&\frac{1}{\pi}\int_0^b h(r, \rho)\left(\frac{1}{\rho - r} - \frac{1}{\rho + r}\right)p_2(\rho)\,\mathrm{d}\rho \\
&\quad + \beta\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho \int_\varepsilon^\infty F_1(r, \rho, \alpha)\,\mathrm{d}\alpha + \beta\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho \int_\varepsilon^\infty F_2(r, \rho, \alpha)\,\mathrm{d}\alpha \\
&\quad + \beta\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho \int_0^\varepsilon \left[-\frac{3r}{2h^3\alpha} - \frac{\alpha r}{2h} + \frac{3}{8}\frac{r\rho^2\alpha}{h^3} + \frac{r^3\alpha}{8h^3} + 0(\alpha^2)\right]\mathrm{d}\alpha \\
&\quad + \beta\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho \int_0^\varepsilon \left[\frac{3r}{2h^3\alpha} + \frac{\alpha r}{2h} - \frac{3}{8}\frac{r\rho^2\alpha}{h^3} - \frac{r^3\alpha}{8h^3} + 0(\alpha^2)\right]\mathrm{d}\alpha \\
&\quad = 0, \qquad (0 \le r < b).
\end{aligned} \tag{28}$$

If we now consider the equilibrium of the layer, i.e.

$$\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho = \int_0^b p_2(\rho)\rho\,\mathrm{d}\rho, \tag{29}$$

(28) becomes

$$\begin{aligned}
&\frac{1}{\pi}\int_0^b h(r, \rho)\left(\frac{1}{\rho - r} - \frac{1}{\rho + r}\right)p_2(\rho)\,\mathrm{d}\rho \\
&\quad + \beta\int_0^b p_2(\rho)\rho\,\mathrm{d}\rho\left[\int_0^\varepsilon\left(\frac{3}{8}\frac{r\rho^2\alpha}{h^3} + 0(\alpha^2)\right)\mathrm{d}\alpha + \int_\varepsilon^\infty F_1(r, \rho, \alpha)\,\mathrm{d}\alpha\right] \\
&\quad + \beta\int_0^a p_1(\rho)\rho\,\mathrm{d}\rho\left[\int_0^\varepsilon\left(-\frac{3}{8}\frac{r\rho^2\alpha}{h^3} + 0(\alpha^2)\right)\mathrm{d}\alpha + \int_\varepsilon^\infty F_2(r, \rho, \alpha)\,\mathrm{d}\alpha\right] \\
&\quad = 0, \qquad (0 \le r < b).
\end{aligned} \tag{30}$$

It is now clear that the Fredholm kernels appearing in (30) are bounded. The integral equations (24) may then be expressed as

$$\begin{aligned}
&\frac{1}{\pi}\int_{-a}^a \frac{p_1(\rho)}{\rho - r}\,\mathrm{d}\rho + \frac{1}{\pi}\int_{-a}^a k(r, \rho)p_1(\rho)\,\mathrm{d}\rho + \gamma\int_{-a}^a k_{11}(r, \rho)p_1(\rho)\,\mathrm{d}\rho \\
&\quad + \gamma\int_{-b}^b k_{12}(r, \rho)p_2(\rho)\,\mathrm{d}\rho = \frac{1}{\gamma_1 + \gamma_3}\frac{\mathrm{d}}{\mathrm{d}r}f(r), \qquad (-a < r < a), \\
&\frac{1}{\pi}\int_{-b}^b \frac{p_2(\rho)}{\rho - r}\,\mathrm{d}\rho + \frac{1}{\pi}\int_{-b}^b k(r, \rho)p_2(\rho)\,\mathrm{d}\rho + \beta\int_{-a}^a k_{21}(r, \rho)p_1(\rho)\,\mathrm{d}\rho \\
&\quad + \beta\int_{-b}^b k_{22}(r, \rho)p_2(\rho)\,\mathrm{d}\rho = 0, \qquad (-b < r < b),
\end{aligned} \tag{31a,b}$$

where $k(r, \rho)$ is given by (25a) and

$$k_{11}(r,\rho) = |\rho|\left[\int_0^{\varepsilon}\left(\frac{3}{8}\frac{r\rho^2\alpha}{h^3} + 0(\alpha^2)\right)\mathrm{d}\alpha + \int_{\varepsilon}^{\infty} F_1(r,\rho,\alpha)\,\mathrm{d}\alpha\right],$$

$$k_{12}(r,\rho) = |\rho|\left[\int_0^{\varepsilon}\left(-\frac{3}{8}\frac{r\rho^2\alpha}{h^3} + 0(\alpha^2)\right)\mathrm{d}\alpha + \int_{\varepsilon}^{\infty} F_2(r,\rho,\alpha)\,\mathrm{d}\alpha\right],$$

$$k_{22}(r,\rho) = k_{11}(r,\rho), \qquad k_{21}(r,\rho) = k_{12}(r,\rho). \tag{32a–d}$$

If the stamp has rounded corners, in the system of singular integral equations (31), in addition to the contact pressures $p_1(r)$ and $p_2(r)$, the radii of the contact areas a and b are also unknown. These two unknown constants a and b are determined from the equilibrium conditions which may be expressed as

$$2\pi\int_0^a p_1(r)r\,\mathrm{d}r = P,$$

$$2\pi\int_0^b p_2(r)r\,\mathrm{d}r = P, \tag{33a,b}$$

where P is the known (compressive) resultant force applied to the stamp away from the contact region $(z = 0,\ 0 \le r < a)$. Without going into any of the details of the solution of the integral equations, from (31) and (33) it is clear that, if $f'(r) = 0$ (i.e. if the stamp is a rigid flat-ended cylinder), then (31) is homogeneous and (31) and (33) can be solved for $p_i(r)/P$, $(i = 1, 2)$ and b uniquely, meaning that the radius of the contact area b is independent of the load amplitude P. On the other hand, if $f'(r) \neq 0$, b and a (if it is unknown) will depend on the actual magnitude of the load P.

3. ON THE SOLUTION OF THE INTEGRAL EQUATIONS

To solve the system of singular integral equations (31) we first normalize the intervals $(-a, a)$ and $(-b, b)$ to be $(-1, 1)$ by defining,

$$\begin{aligned}
&r = ax, \qquad \rho = at, \qquad (-a < (r,\rho) < a),\\
&r = bx, \qquad \rho = bt, \qquad (-b < (r,\rho) < b),\\
&p_1(\rho) = p_1(at) = g_1(t), \qquad (-1 < t < 1),\\
&p_2(\rho) = p_2(bt) = g_2(t), \qquad (-1 < t < 1),\\
&\frac{1}{\gamma_1+\gamma_3}\frac{\mathrm{d}}{\mathrm{d}r}f(r) = F(x), \qquad (-1 < x < 1).
\end{aligned} \tag{34}$$

Thus, (31) and (33) become

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g_1(t)}{t-x}\,\mathrm{d}t + \frac{a}{\pi}\int_{-1}^{1}k(ax,at)g_1(t)\,\mathrm{d}t + a\gamma\int_{-1}^{1}k_{11}(ax,at)g_1(t)\,\mathrm{d}t$$
$$+ b\gamma\int_{-1}^{1}k_{12}(ax,bt)g_2(t)\,\mathrm{d}t = F(x), \qquad (|x| < 1),$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g_2(t)}{t-x}\,\mathrm{d}t + \frac{b}{\pi}\int_{-1}^{1}k(bx,bt)g_2(t)\,\mathrm{d}t + a\beta\int_{-1}^{1}k_{21}(bx,at)g_1(t)\,\mathrm{d}t$$
$$+ b\beta\int_{-1}^{1}k_{22}(bx,bt)g_2(t)\,\mathrm{d}t = 0, \qquad (|x| < 1); \tag{35a,b}$$

$$2\pi a^2 \int_0^1 g_1(x)x\,\mathrm{d}x = P,$$

$$2\pi b^2 \int_0^1 g_2(x)x\,\mathrm{d}x = P. \tag{36a,b}$$

(35) is an ordinary system of singular integral equations. Hence, referring to[19], the solution is of the following form

$$g_1(t) = (1 - t^2)^{\mp 1/2} G_1(t),$$
$$g_2(t) = (1 - t^2)^{\mp 1/2} G_2(t), \tag{37a,b}$$

where G_1 and G_2 are bounded in $(-1 \le t \le 1)$. Since the contact between the layer and the subspace is always "smooth", $g_2(\mp 1)$ must be bounded (and necessarily zero). Therefore,

$$g_2(t) = (1 - t^2)^{1/2} G_2(t), \tag{38}$$

and the index of (35b) is always -1. Similarly if body 3, the stamp, has rounded corners, contact between the stamp and the layer will also be smooth, and in (37a) $+1/2$ will be the correct exponent. On the other hand if the stamp is a rigid cylinder with a circular sharp edge along $r = a$, then the pressure p_1 will have an integrable singularity at $r = a$, and in (37a) $-1/2$ will be the correct exponent. In either case, in the limiting case of $h \to \infty$ the functions $F_i(r, \rho, \alpha)$, $(i = 1, 2)$ (see (22)), and consequently the kernels k_{ij} will vanish and the integral equations will be uncoupled. The first becomes the integral equation for a stamp (body 3) on a half space (body 1). Since the second equation is homogeneous with an index of -1, as $h \to \infty$ its solution p_2 will tend to zero. Even though the technique described in [20] may be used to solve the system of integral equations (35) in a straight-forward manner, for various stamp geometries there are certain differences in the approach which are summarized below:

(a) *Elastic or rigid stamp with a rounded profile*

In this case both a and b are unknown. As in all stamp problems of this type, since the problem is highly nonlinear in a it is solved in an inverse manner, i.e. by assuming that the contact radius a rather than the resultant force P is prescribed. For each given a, P is then calculated from (36a). Similarly, since the problem is also highly nonlinear in b, it can only be determined by some kind of interpolation. From (36) defining

$$H(b) = a^2 \int_0^1 g_1(x)x\,\mathrm{d}x - b^2 \int_0^1 g_2(x)x\,\mathrm{d}x \tag{39}$$

it is seen that for the correct value of b the function $H(b)$ is zero. Hence for a given radius a and for $b = b_1, b_2, \ldots$, solving† (35) for g_1 and g_2, and evaluating $H(b_i)$, $(i = 1, 2, \ldots)$ from (39), an interpolation scheme may be established to determine b.

(b) *Flat-ended rigid cylindrical stamp*

In this case a is known and we have

$$F(x) = 0, \qquad g_1(t) = (1 - t^2)^{-1/2} G_1(t). \tag{40a,b}$$

† Note that in this case, the index of both equations in (35) is -1, therefore g_1, g_2 may be calculated uniquely once a and b are specified.

If (35) and (36) are divided by P, it is seen that the unknown functions of the problem are

$$\phi_1(x) = g_1(x)/P, \qquad \phi_2(x) = g_2(x)/P. \tag{41a,b}$$

The index of (35a) is 1 and that of (35b) is -1. Hence the solution will contain one arbitrary constant which is determined from

$$2\pi a^2 \int_0^1 \phi_1(x)x \, dx = 1. \tag{42}$$

For this problem, the interpolation scheme to determine b may be established by considering the function

$$M(b) = 2\pi b^2 \int_0^1 \phi_2(x)x \, dx - 1, \tag{43}$$

$M(b) = 0$ corresponding to the correct value of b, which clearly is independent of P.

(c) *Rigid cylindrical stamp with a sharp edge and an arbitrary end profile*

In this problem, a and $F(x)$ are known, (40b) is still valid, the index of (35a) is 1, (36a) is used to determine the related arbitrary constant, and the interpolation scheme to determine the unknown constant b may be set up by considering the function

$$\begin{aligned} N(b) &= P - 2\pi b^2 \int_0^1 g_2(x)x \, dx \\ &= 2\pi a^2 \int_0^1 g_1(x)x \, dx - 2\pi b^2 \int_0^1 g_2(x)x \, dx. \end{aligned} \tag{44}$$

Unlike the flat-ended cylinder, for this stamp geometry the radius b of the contact area between the layer and the subspace is dependent on the load magnitude P.

4. EXAMPLES

As examples two basic stamp geometries will be considered. The first is a spherical stamp (or a paraboloid of rotation) with a radius of curvature R for which we have (see (4b), (19a), (34), and (35a))

$$f(r) = \frac{1}{2R}(a^2 - r^2), \qquad F(x) = -\frac{a}{R(\gamma_1 + \gamma_3)}x, \tag{45a,b}$$

where a is the radius of the stamp-layer contact area. The second is a flat-ended rigid cylindrical stamp of radius a for which

$$F(x) = 0. \tag{46}$$

In the numerical analysis the layer thickness h will be used as the length unit.

The results for the flat-ended stamp obtained from (35) and (41)–(43) with $M(b) = 0$ are given in Figs. 2–7. Figure 2 shows the layer-to-subspace contact radius b as a function of the bimaterial constant β defined by (21) for various values of a. For $a/h = 0{\cdot}01$ the result given in Fig. 2 is indistinguishable from that given in[11] for a concentrated force acting at $r = 0$, $z = 0$. An interesting feature of the flat stamp problem shown in Fig. 2 is that, if a/h is sufficiently large, for small values of β (i.e. for large values of μ_1/μ_2) a separation of the

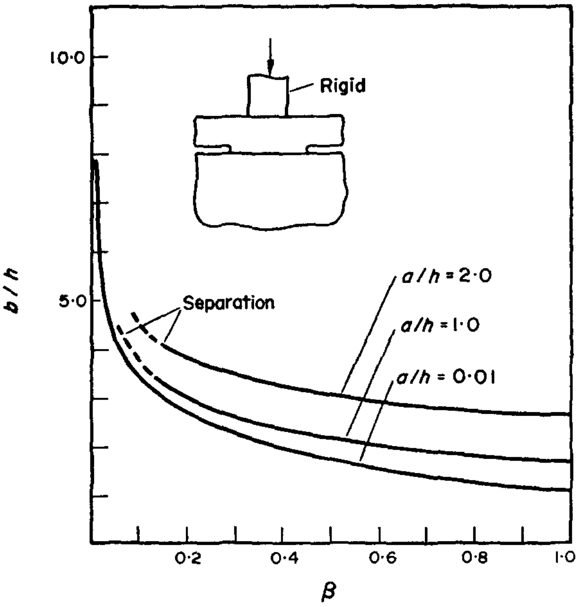

Fig. 2. Layer-half space contact radius b for loading by a flat-ended rigid stamp of diameter $2a$.

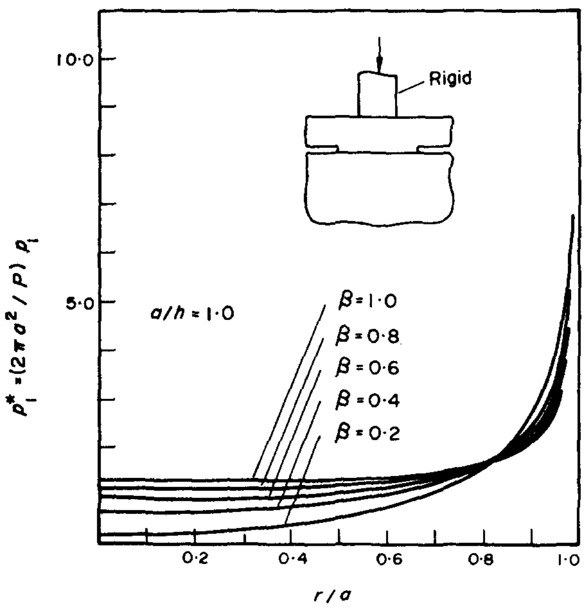

Fig. 3. Contact pressure between the flat stamp and the layer for $a/h = 1.0$ and for various layer-half space material combinations β.

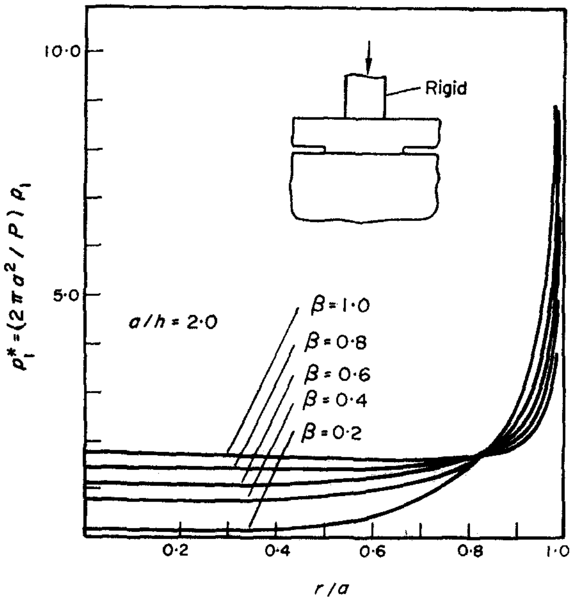

Fig. 4. Contact pressure between the flat stamp and the layer for $a/h = 2{\cdot}0$.

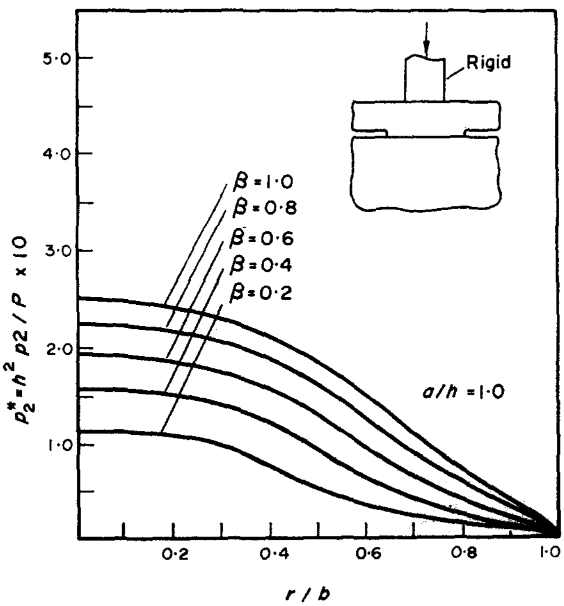

Fig. 5. Contact pressure between the layer and the half space for $a/h = 1.0$ and for loading by a flat stamp.

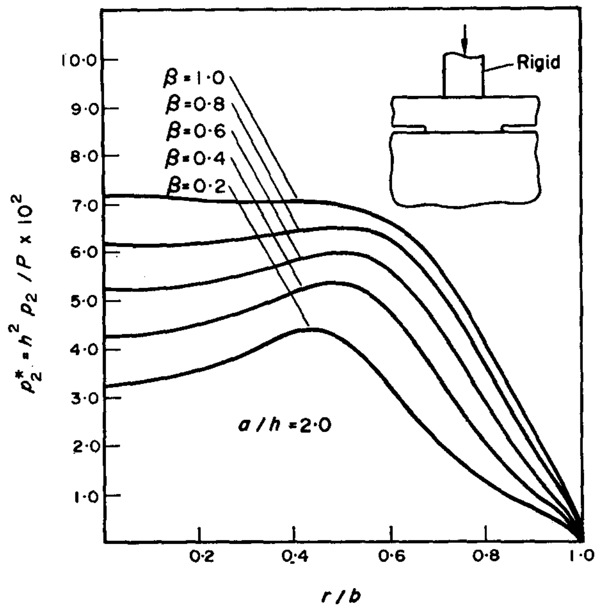

Fig. 6. Contact pressure between the layer and the half space for $a/h = 2{\cdot}0$ and for loading by a flat stamp.

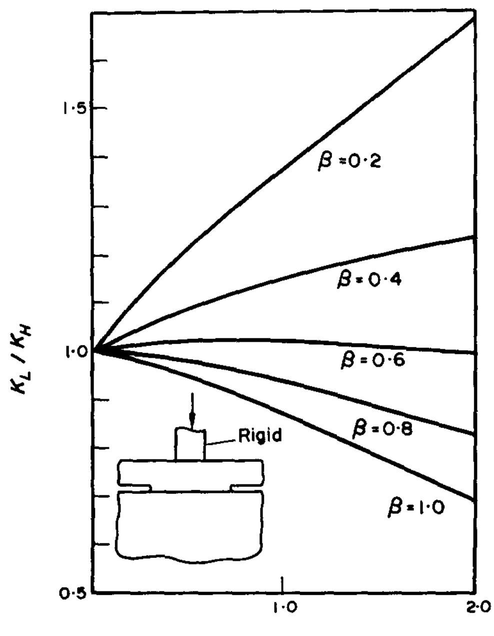

Fig. 7. The stress intensity factor ratio for the flat-ended rigid stamp.

contacting surfaces of the stamp and the layer may take place around $r = 0$, that is the pressure $p_1(r)$ may become negative. This trend may also be seen from Figs. 3 and 4 where the distribution of the normalized contact pressure $p_1(r)$ is shown for various values of β. If β is further decreased, around $r = 0$ $p_1(r)$ becomes zero and changes sign. The solution given here, of course, would not be valid for this case. This separation problem is one of receding contact on both sides of the layer and can be formulated and solved by using a technique similar to that described in this paper. The value of β at the initiation of separation is shown by dashed lines in Fig. 2. Figures 5 and 6 show the contact pressure between the layer and the subspace. For $a/h \to 0$, the pressure under the stamp approaches that for a rigid stamp on an elastic half space[3] given by

$$p_1(r) = \frac{P}{2\pi a} \frac{1}{\sqrt{a^2 - r^2}}. \tag{47}$$

In the present problem p_1 is given by (34) and (40), i.e.

$$p_1(r) = G_1(r/a)/(1 - r^2/a^2)^{1/2}. \tag{48}$$

Thus the contact pressure has an integrable singularity at $r = a$ and the strength of the singularity or the stress intensity factor may be expressed as

$$K = \lim_{r \to a} \sqrt{2(a - r)}p_1(r). \tag{49}$$

Figure 7 shows the ratio of the stress intensity factor K_L for the layer problem to that for the half space given by

$$K_H = P/(2\pi a^{3/2}). \tag{50}$$

The figure shows that for small values of β ($\sim\beta < 0{\cdot}6$), $K_L/K_H > 1$. The same trend is also observed in crack problems of two elastic half spaces bonded through an elastic layer containing a crack. In all such problems, generally if the stiffness of the half spaces is smaller than that of the layer, the stress intensity is greater than that corresponding to a homogeneous infinite space containing a penny-shaped crack[21]. It shoud be noted that if the interface condition between the layer and the half space were to be perfect adhesion, then for $\beta = 0{\cdot}5$ the problem would be that of a rigid stamp on a half space for which $K_L/K_H = 1$. On the other hand Fig. 7 shows that for $\beta = 0{\cdot}5$ $K_L/K_H > 1$. Thus the removal of the tangential constraint on the interface corresponds to a reduction in the modulus of the half space for the perfect adhesion case.

In solving the spherical stamp problem we observe that the additional length parameter R, the radius of the sphere, enters into the analysis only through $F(x)$, the right hand side of (35a), (see (45b)). Thus defining

$$p_0 = \frac{a}{R(\gamma_1 + \gamma_3)}, \qquad p_1{}^*(r) = p_1(r)/p_0 = g_1(x)/p_0 = g_1{}^*(x),$$

$$P^* = P/(p_0 a^2) = 2\pi \int_0^1 g_1{}^*(x) x \, dx, \qquad H^*(b) = H(b)/p_0,$$

$$p_2(r)/p_0 = g_2(x)/p_0 = n_2{}^*(x), \tag{51}$$

it is seen that for given a, h, β and γ, (35) can be solved for $g_1{}^*(x)$, $n_2{}^*(x)$, b (with $H^*(b) = 0$), and P^*. Then for a given R, p_0, p_1, p_2 and P can be evaluated from (51). The results for a rigid spherical stamp ($\gamma_3 = 0$) are given in Figs. 8–13. In order to compare the layer–subspace pressures obtained from two different stamp geometries, for p_2 the same normalization pressure (P/h^2) is used in both examples. Note that the quantity $p_2{}^*(r)$ given in the figures

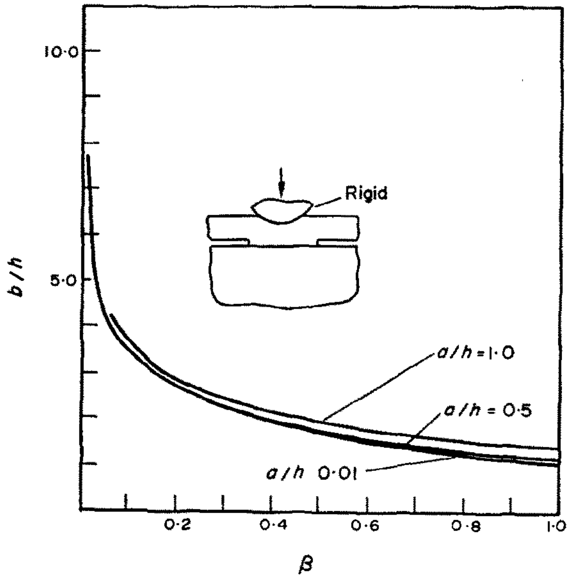

Fig. 8. Layer-half space contact radius b for loading by a rigid spherical stamp.

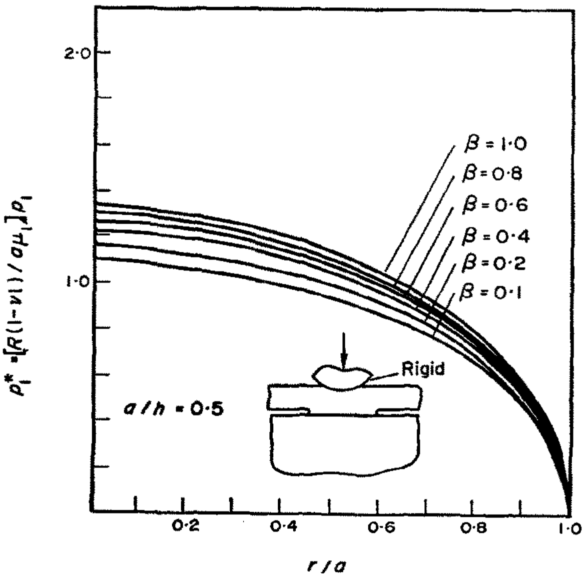

Fig. 9. Contact pressure between the rigid spherical stamp and the layer for $a/h = 0{\cdot}5$.

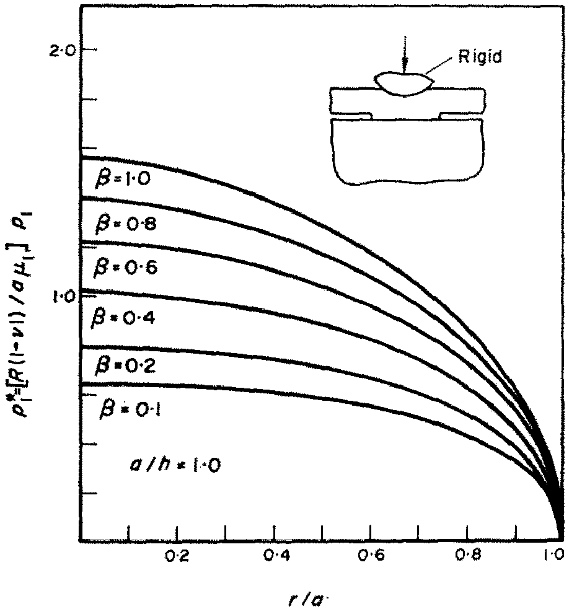

Fig. 10. Contact pressure between the rigid spherical stamp and the layer for $a/h = 1{\cdot}0$.

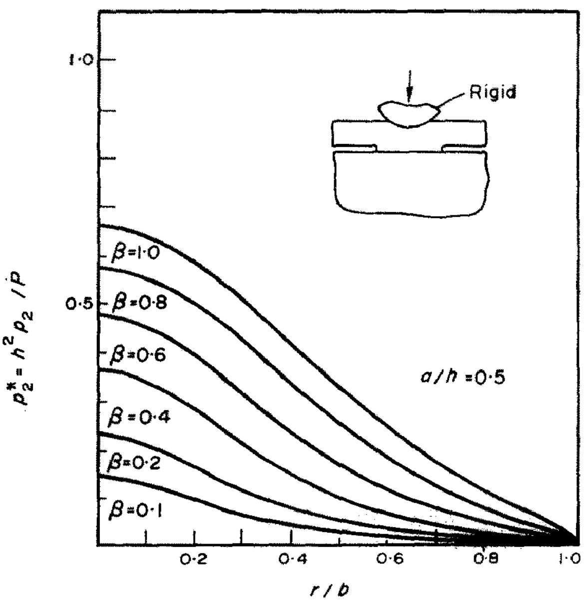

Fig. 11. Contact pressure between the layer and the half space for $a/h = 0{\cdot}5$ and for loading by a rigid sphere.

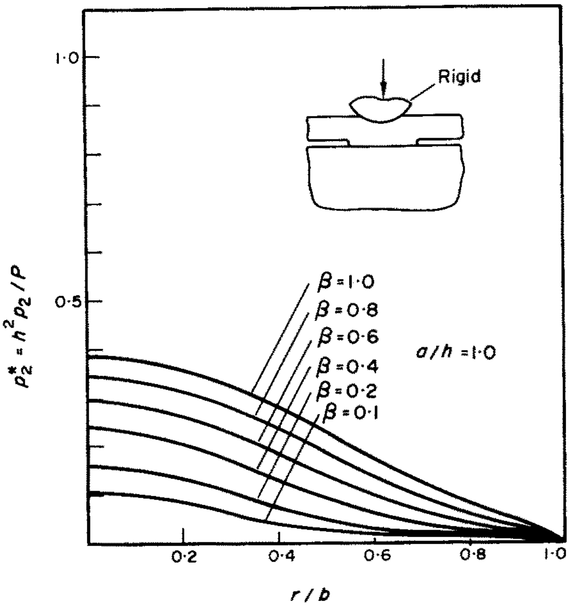

Fig. 12. Contact pressure between the layer and the half space for $a/h = 1{\cdot}0$ and for loading by a rigid sphere.

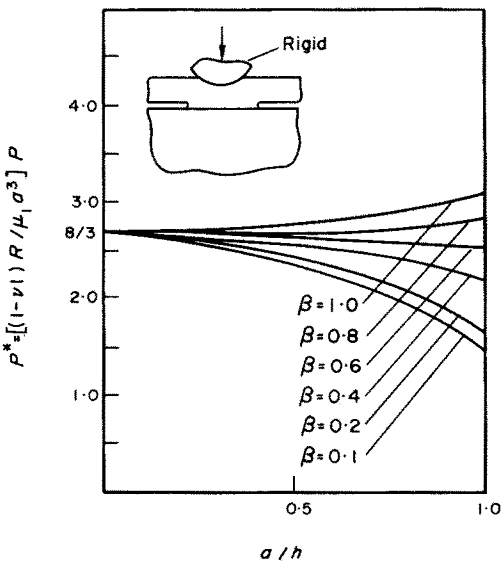

Fig. 13. Total transmitted force P vs the stamp-layer contact radius a for various layer-half space material combinations β and for loading by a rigid spherical stamp.

may be obtained from

$$p_2^*(r) = \frac{p_2(r)}{P/h^2} = \frac{n_2^*(r/a)}{P^* a^2/h^2}. \tag{52}$$

For $a/h = 0{\cdot}01$, b and the contact pressure $p_2(r)$ obtained from the spherical and the flat-ended stamp problems are nearly identical and are the same as that obtained in[11]. Figure 13 shows the results for P^* which is used (in an inverse way) to determine the applied load P for given a, h, R and β representing the properties of the materials. In the limiting case as $a/h \to 0$ the results regarding the stamp-layer contact pressure (see (47)) reduce to the half space solution for which[1]

$$P = \frac{8}{3}\frac{a^3}{(\gamma_1 + \gamma_3)R} = \frac{8}{3}a^2 p_0. \tag{53}$$

Comparing (51) and (53) it is seen that for $a/h \to 0$ $P^* \to 8/3$ which is seen to be the value to which P^* vs a/h curves converge for all values of β in Fig. 13.

Figures 14–17 show the results for an elastic spherical stamp. For $a/h = 0{\cdot}01$ the results are indistinguishable from rigid stamp results. For larger values of a/h, however, the difference may be considerable. For example, as seen from Fig. 17, for a fixed value of a/h the total transmitted force P appears to be nearly independent of β for low values (up to 0·3) of γ (i.e. for a stamp with relatively low stiffness), whereas for a rigid stamp P varies over a relatively wide range as β varies between 0 and 1.

In conclusion it could be stated that the technique used in this paper for formulating the problem in terms of a system of singular integral equations can be used without major modification to solve similar contact problems for multilayered materials, for more complex contact geometries, and for problems in which body forces (such as gravity) may no longer be negligible.

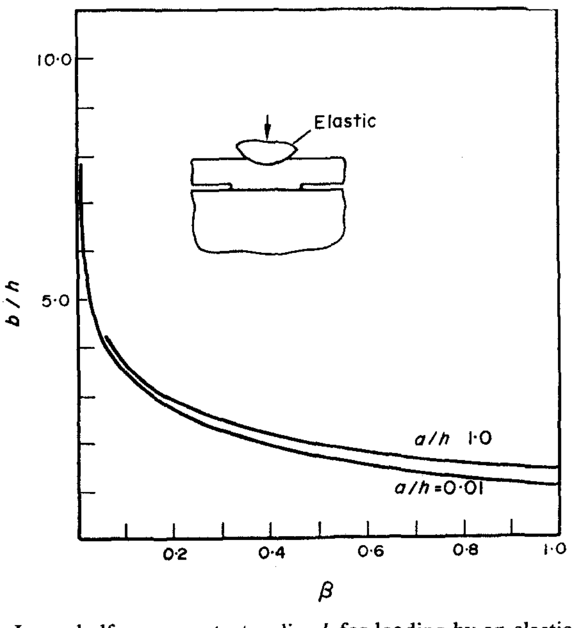

Fig. 14. Layer-half space contact radius b for loading by an elastic stamp.

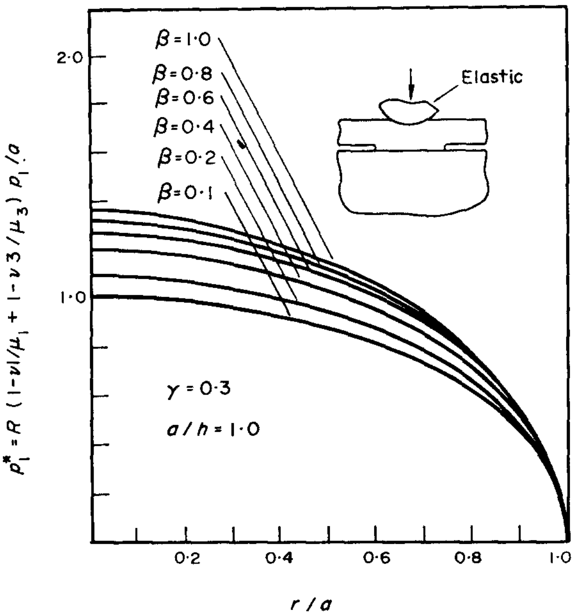

Fig. 15. Contact pressure between the spherical elastic stamp and the layer for $a/h = 1{\cdot}0$ and $\gamma = 0{\cdot}3$.

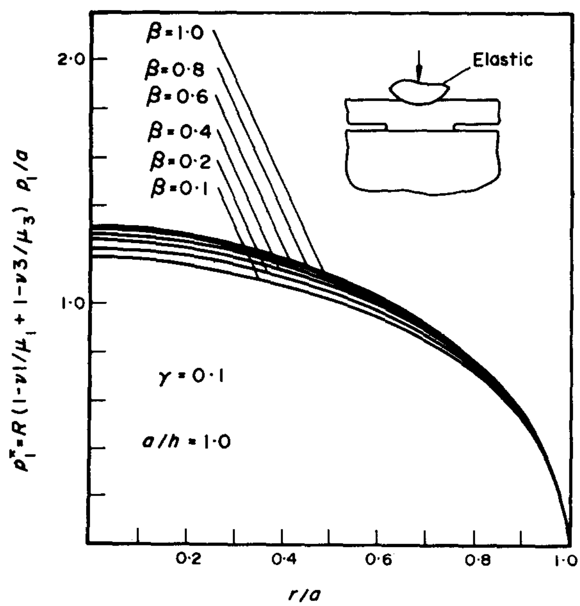

Fig. 16. Contact pressure between the spherical elastic stamp and the layer for $a/h = 1{\cdot}0$ and $\gamma = 0{\cdot}1$.

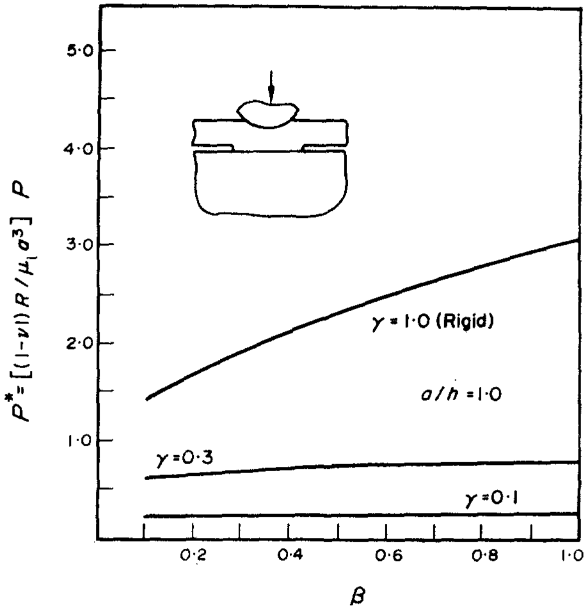

Fig. 17. Total transmitted force P vs the layer-subspace material constant β for various values of stamp-layer material constant γ, and for $a/h = 1{\cdot}0$ in loading through an elastic sphere.

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APPENDIX

The functions $m_{ij}(\alpha)$ (see equation 15):

$$m_{11}(\alpha) = \frac{(\kappa_1 - 1)(e^{-2\alpha h} - 2\alpha h - 1) + 4\alpha^2 h_2}{4\mu_1 \alpha(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{12}(\alpha) = \frac{(\kappa_1 - 1)(e^{\alpha h} - e^{-\alpha h}) + 2\alpha h(\kappa_1 e^{-\alpha h} - e^{\alpha h})}{4\mu_1 \alpha(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{21}(\alpha) = \frac{2\alpha h - e^{-2\alpha h} + 1}{2\mu_1(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{22}(\alpha) = -\frac{(2\alpha + e^{2ah} - 1)e^{-ah}}{2\mu_1(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{31}(\alpha) = \frac{1}{4\mu_1 \alpha}\left[-(\kappa_1 - 1) + \frac{(\kappa_1 - 1)(4\alpha^2 h^2 - e^{-2\alpha h} + 2\alpha h + 1) + 4\alpha^2 h^2}{4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2}\right],$$

$$m_{32}(\alpha) = \frac{(\kappa_1 - 1)(e^{-\alpha h} - e^{\alpha h}) - 2\alpha h(\kappa_1 e^{\alpha h} - e^{-\alpha h})}{4\mu_1 \alpha(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{41}(\alpha) = \frac{1}{2\mu_1}\left[-1 + \frac{4\alpha^2 h^2 + 2\alpha h - e^{-2\alpha h} + 1}{4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2}\right],$$

$$m_{42}(\alpha) = \frac{(e^{-2\alpha h} - 2\alpha h - 1)e^{\alpha h}}{2\mu_1(4\alpha^2 h^2 - e^{-2\alpha h} - e^{2\alpha h} + 2)},$$

$$m_{51}(\alpha) = 0, \qquad m_{52}(\alpha) = -\frac{(1 + \alpha h - 2\nu_2)e^{\alpha h}}{2\mu_2 \alpha},$$

$$m_{61}(\alpha) = 0, \qquad m_{62}(\alpha) = -\frac{e^{\alpha h}}{2\mu_2},$$

$$m_{71}(\alpha) = -\frac{1 - 2\nu_3}{2\mu_3 \alpha}, \qquad m_{72}(\alpha) = 0,$$

$$m_{81}(\alpha) = \frac{1}{2\mu_3}, \qquad m_{82}(\alpha) = 0.$$

Абстракт—Рассматривается общая проблема осесимметричного двойного контакта эластичного слоя прижатого к полупространству эластичным пестиком. Проблема решается предположением, что три материала имеют различные эластичные свойства, что контакт по поверхности раздела лишен трения и, что по поверхности раздела могут передаваться только нормальные сдавливающие тяговые усилия. Местный радиус кривизны пестика по сравнению с радиусом контакта пестика-слоя более большой. Проблема приводится к системе сингулярных интегральных уравнений в которых давление контакта является неизвестной функцией. Достигли решения и даются численные результаты трех геометрических соображений для пестика, а именно, для жестких и эластичных сферических пестиков и для жесткого цилиндрического пестика с плоскими концами. По результатам видно, что в случае жесткого цилиндрического пестика с плоскими концами радиус b — площадь контакта между слоем и веществом зависит от величины P — общей переданной нагрузки, и во всех других случаях b будет зависеть от P.